

A NOTE ON THE q -ANALOGUE OF p -ADIC log-GAMMA FUNCTION

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ABSTRACT. In this paper we prove that the q -analogue of Euler numbers occur in the coefficients of some stirring type series for the p -adic analytic q -log-gamma function.

§1. Introduction

Let p be a fixed odd prime number. Throughout this paper \mathbb{Z} , \mathbb{Q} , \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will respectively denote the ring of rational integers, the field of rational numbers, the ring p -adic rational integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let v_p be the normalized exponential valuation of \mathbb{C}_p such that $|p|_p = p^{-v_p(p)} = p^{-1}$. If $q \in \mathbb{C}_p$, we normally assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

Hence, $\lim_{q \rightarrow 1} [x]_q = 1$, for any x with $|x|_p \leq 1$ in the present p -adic case.

For $d(= \text{odd})$ a fixed positive integer with $(p, d) = 1$, let

$$\begin{aligned} X &= X_d = \varprojlim_N \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 \leq a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, cf.[1-14].

In [3-7, 16], it is known that

$$\mu_{-q}(a + dp^N \mathbb{Z}_p) = (1 + q) \frac{(-1)^a q^a}{1 + q^{dp^N}} = \frac{(-q)^a}{[dp^N]_{-q}},$$

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is distribution on X for $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. This distribution yields an integral as follows:

$$(1) \quad I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ for } f \in UD(\mathbb{Z}_p),$$

which has a sense as we see readily that the limit is convergent.

For $q = 1$, we have fermionic p -adic integral on \mathbb{Z}_p as follows:

$$I_{-1} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.$$

In view of notation, I_{-1} can be written symbolically as $I_{-1}(f) = \lim_{q \rightarrow -1} I_q(f)$, where $I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x$, see [3].

As the formula of the stirling asymptotic series, it was well known that

$$\log \left(\frac{\Gamma(x+1)}{\sqrt{2\pi}} \right) = \left(x - \frac{1}{2}\right) \log x - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{B_{n+1}}{x^n}, \text{ cf. [15],}$$

where B_n are called the n -th Bernoulli numbers.

The purpose of this paper is to give the new formula of the p -adic q -analogue of $\log \left(\frac{\Gamma(x+1)}{\sqrt{2\pi}} \right)$, which is related to q -Euler numbers. That is, we prove that the q -analogue of Euler numbers occur in the coefficients of some stirling type series for p -adic analytic q -log-gamma functions.

§2. p -adic q -log-gamma function

Let us include some remarks about the factorial function, we define $0! = 1$ and may compute further values by the relation $(n+1)! = n!(n+1)$. For large n the function is very large. A convenient approximation for large n is the stirling formula:

$$(1-2) \quad n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad (e = 2.718 \dots), \text{ cf. [15],}$$

where \sim means that the ratio of two sides of (1-2) approaches 1 as n approaches infinity.

From (1-2) we can derive

$$(2) \quad \log \left(\Gamma(x+1)/\sqrt{2\pi} \right) = (x + B_1) \log x - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{B_{n+1}}{x^n}, \text{ cf. [4, 15],}$$

where B_n are called the n -th Bernoulli numbers.

For any non-negative integer m , we define the q -Euler polynomials as follows:

$$(3) \quad \int_{\mathbb{Z}_p} [x+y]_q^m d\mu_{-q}(y) = E_{m,q}(x) = [2]_q \left(\frac{1}{1-q} \right)^m \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{q^x}{1+q^{i+1}}.$$

From (3), we can also derive q -Euler numbers, $E_{n,q}$, as $E_{n,q}(0) = E_{n,q}$. Note that $\lim_{q \rightarrow 1} E_{n,q} = E_n$, where E_n are ordinary Euler numbers which are defined by $\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$. By the simple calculation, it is easy to show that

$$(4) \quad ((1+x) \log(1+x))' = 1 + \log(1+x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n,$$

where $((1+x) \log(1+x))' = \frac{d}{dx} ((1+x) \log(1+x))$.

From (4) we derive

$$(5) \quad (1+x) \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x + c, \text{ where } c \text{ is constant.}$$

If we take $x = 0$, then we have $c = 0$. By (3) and (4), we easily see that

$$(6) \quad (1+x) \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x.$$

We now consider p -adic locally analytic function $G_{p,q}(x)$ on $\mathbb{C}_p \setminus \mathbb{Z}_p$ by

$$(7) \quad G_{p,q}(x) = \int_{\mathbb{Z}_p} [x+z]_q (\log[x+z]_q - 1) d\mu_{-q}(z).$$

From (1) we can easily derive

$$(8) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1 \text{ is translation with } f_1(x) = f(x+1).$$

By (7) and (8), we easily see that

$$qG_{p,q}(x+1) + G_{p,q}(x) = [2]_q ([x]_q (\log[x]_q - 1)).$$

It is easy to see that

$$(9) \quad [x+z]_q = \frac{1-q^{x+z}}{1-q} = \frac{1-q^x + q^x(1-q^z)}{1-q} = [x]_q + q^x [z]_q.$$

By (6) and (9) we see that

$$\begin{aligned}
(10) \quad & [x+z]_q (\log[x+z]_q - 1) \\
&= [z]_q + [x]_q \sum_{n=1}^{\infty} \frac{(-q^x)^{n+1}}{n(n+1)} \frac{[z]_q^{n+1}}{[x]_q^{n+1}} + ([x]_q + q^x [z]_q) \log[x]_q - ([x]_q + [z]_q).
\end{aligned}$$

From (3), (7) and (10), we note that

$$G_{p,q}(x) = ([x]_q + q^x E_{1,q}) \log[x]_q - [x]_q + \sum_{n=1}^{\infty} \frac{(-q^x)^{n+1}}{n(n+1)} \frac{1}{[x]_q^n} E_{n+1,q}.$$

Therefore we obtain the following:

Theorem A. *For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, we have*

$$(11) \quad G_{p,q}(x) = ([x]_q - q^x \frac{1}{[2]_{q^2}}) \log[x]_q - [x]_q + \sum_{n=1}^{\infty} \frac{(-q^x)^{n+1}}{n(n+1)} \frac{1}{[x]_q^n} E_{n+1,q},$$

and

$$(12) \quad qG_{p,q}(x+1) + G_{p,q}(x) = [2]_q ([x]_q (\log[x]_q - 1)).$$

Remark. The above Theorem A seems to be the p -adic q -analogue of $\log \frac{\Gamma(x+1)}{\sqrt{2\pi}}$, which is related to q -Euler numbers. In [4], q -Bernoulli numbers defined by

$$\int_{\mathbb{Z}_p} q^{-x} [x]_q^n d\mu_q(x) = \beta_{n,q}.$$

For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, we consider the p -adic q -log-gamma function as follows:

$$(13) \quad T_{p,q}(x) = \int_{\mathbb{Z}_p} q^{-y-x} [x+y]_q (\log[x+y]_q - 1) d\mu_q(y).$$

From (13) and (6) it seems to be derived the following interesting formula:

$$T_{p,q}(x) = (q^{-x} [x]_q \beta_{0,q} + \beta_{1,q}) \log[x]_q - q^{-x} [x]_q \beta_{0,q} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{nx}}{n(n+1)} \frac{\beta_{n+1,q}}{[x]_q^n}.$$

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